定数変化法　次の微分方程式を解きなさい。21～24については初期条件を満たす解を求めなさい。（M. Tennenbaum & H. Pollard: *Ordinary Differential Equations* (1963, 1985)より）

Find the general solution of each of the following.

1. $xy' + y = x^3$.
2. $y' + ay = b$.
3. $xy' + y = y^2 \log x$.
4. $\dfrac{dx}{dy} + 2yx = e^{-y^2}$. *Hint.* Consider x as the dependent variable.
5. $\dfrac{dr}{d\theta} = (r + e^{-\theta}) \tan \theta$.
6. $\dfrac{dy}{dx} - \dfrac{2xy}{x^2 + 1} = 1$.
7. $y' + y = xy^3$.
8. $(1 - x^3) \dfrac{dy}{dx} - 2(1 + x)y = y^{5/2}$.
9. $\tan \theta \dfrac{dr}{d\theta} - r = \tan^2 \theta$.
10. $L \dfrac{di}{dt} + Ri = E \sin kt$. (This is the equation of a simple electric circuit containing an inductor, a resistor, and an applied electromotive force. For the meaning of these terms and for a more complete discussion of electric circuits, see Lessons 30 and 33C.)

11. $y' + 2y = 3e^{-2x}$.
12. $y' + 2y = \frac{3}{4}e^{-2x}$.
13. $y' + 2y = \sin x$.
14. $y' + y \cos x = e^{2x}$.
15. $y' + y \cos x = \frac{1}{2} \sin 2x$.
16. $xy' + y = x \sin x$.
17. $xy' - y = x^2 \sin x$.
18. $xy' + xy^2 - y = 0$.
19. $xy' - y(2y \log x - 1) = 0$.
20. $x^2(x - 1)y' - y^2 - x(x - 2)y = 0$.

Find a particular solution of each of the differential equations 21–24.

21. $y' - y = e^x, \quad y(0) = 1$.
22. $y' + \dfrac{1}{x} y = \dfrac{y^2}{x}, \quad y(-1) = 1$.
23. $2 \cos x\, dy = (y \sin x - y^3)\, dx, \quad y(0) = 1$.
24. $(x - \sin y)\, dy + \tan y\, dx = 0, \quad y(1) = \pi/6$.
25. The differential equation

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2, \quad f_2(x) \not\equiv 0, \tag{11.6}$$

is called a **Riccati equation.** If $y_1(x)$ is a particular solution of this equation, show that the substitution

$$y = y_1 + \frac{1}{u}, \qquad y' = y_1' - \frac{1}{u^2} u', \tag{11.61}$$

will transform the equation into the first order linear equation

$$u' + [f_1(x) + 2f_2(x)y_1]u = -f_2(x). \tag{11.62}$$

Hint. Since y_1 is a particular solution of the given equation, $y_1' = f_0(x) + f_1(x)y_1 + f_2(x)y_1^2$.

With the aid of problem 25 above, find the general solution of each of the following Riccati equations.

26. $y' = x^3 + \frac{2}{x}y - \frac{1}{x}y^2, \quad y_1(x) = -x^2.$

27. $y' = 2\tan x \sec x - y^2 \sin x, \quad y_1(x) = \sec x.$

28. $y' = \frac{1}{x^2} - \frac{y}{x} - y^2, \quad y_1(x) = \frac{1}{x}\cdot$

29. $y' = 1 + \frac{y}{x} - \frac{y^2}{x^2}, \quad y_1(x) = x.$

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1. $4xy = x^4 + c.$

2. $y = \frac{b}{a} + ce^{-ax}.$

3. $y \log x + y + cxy = 1.$

4. $x = e^{-y^2}(y + c).$

5. $2r = c \sec\theta - e^{-\theta}(\tan\theta + 1).$

6. $y = (\operatorname{Arc\,tan} x + c)(x^2 + 1).$

7. $\frac{1}{y^2} = ce^{2x} + x + \frac{1}{2}\cdot$

8. $y^{-3/2} = -\frac{3}{4(1 + x + x^2)} + \frac{c(1 - x)^2}{1 + x + x^2}\cdot$

9. $r = \sin\theta[\log(\sec\theta + \tan\theta)] + c\sin\theta.$

10. $i = ce^{-Rt/L} + \frac{E(R\sin kt - kL\cos kt)}{R^2 + k^2L^2}\cdot$

11. $y = 3xe^{-2x} + ce^{-2x}.$

12. $y = \frac{3}{4}xe^{-2x} + ce^{-2x}.$

13. $y = \frac{1}{5}(2\sin x - \cos x) + ce^{-2x}.$

14. $y = e^{-\sin x}\left(c + \int e^{2x+\sin x}\,dx\right)\cdot$

15. $y = \sin x - 1 + ce^{-\sin x}.$

16. $y = \frac{\sin x}{x} - \cos x + \frac{c}{x}\cdot$

17. $y = x(c - \cos x).$

18. $y = \frac{2x}{x^2 + c}, \quad y = 0.$

19. $1 - 2y(1 + \log x) = cxy.$

20. $y = \frac{x^2}{(x - 1)c + 1}\cdot$

21. $y = e^x(x + 1).$

22. $y = 1.$

23. $\sec x = y^2(\tan x + 1).$

24. $8x \sin y = 4\sin^2 y + 3.$

26. $u' + \left(\frac{2}{x} + 2x\right)u = \frac{1}{x}, \qquad y = -x^2 + \frac{2x^2e^{x^2}}{e^{x^2} + c}\cdot$

27. $u' - 2u\tan x = \sin x, \qquad y = \frac{1}{\cos x} + \frac{3\cos^2 x}{c - \cos^3 x}\cdot$

28. $u' - \frac{3}{x}u = 1, \qquad y = \frac{1}{x} + \frac{2}{cx^3 - x}\cdot$

29. $u' - \frac{1}{x}u = \frac{1}{x^2}, \qquad y = x + \frac{2x}{cx^2 - 1}\cdot$