2階定数係数線形非斉次微分方程式　3〜27 の微分方程式の一般解を求めなさい。28〜33 については境界条件を満たす解を求めなさい。(M. Tennenbaum & H. Pollard: *Ordinary Differential Equations* (1963, 1985) より)

EXERCISE 21

1. Prove that any term which is in the complementary function y_c need not be included in the trial solution y_p. (*Hint.* Show that the coefficients of this term will always add to zero.)

2. Prove that if $F(x)$ is a function with a finite number of linearly independent derivatives, i.e., if $F^{(n)}(x)$, $F^{(n-1)}(x)$, $\cdots$, $F'(x)$, $F(x)$ are linearly independent functions, where n is a finite number, then $F(x)$ consists only of such terms as a, x^k, e^{ax}, $\sin ax$, $\cos ax$, and combinations of such terms, where a is a constant and k is a positive integer. *Hint.* Set the linear combination of these functions equal to zero, i.e., set

$$C_nF^{(n)}(x) + C_{n-1}F^{(n-1)}(x) + \cdots + C_1F'(x) + C_0F(x) = 0,$$

where the C's are not all zero, and then show, by Lesson 20, that the only functions $F(x)$ that can satisfy this equation are those stated.

Find the general solution of each of the following equations.

3. $y'' + 3y' + 2y = 4.$

4. $y'' + 3y' + 2y = 12e^x.$

5. $y'' + 3y' + 2y = e^{ix}.$

6. $y'' + 3y' + 2y = \sin x.$

7. $y'' + 3y' + 2y = \cos x.$

8. $y'' + 3y' + 2y = 8 + 6e^x + 2\sin x.$

9. $y'' + y' + y = x^2.$

10. $y'' - 2y' - 8y = 9xe^x + 10e^{-x}.$

11. $y'' - 3y' = 2e^{2x}\sin x.$

12. $y^{(4)} - 2y'' + y = x - \sin x.$

13. $y'' + y' = x^2 + 2x.$

14. $y'' + y' = x + \sin 2x.$

15. $y'' + y = 4x\sin x.$

16. $y'' + 4y = x\sin 2x.$

17. $y'' + 2y' + y = x^2e^{-x}.$

18. $y''' + 3y'' + 3y' + y = 2e^{-x} - x^2e^{-x}.$

19. $y'' + 3y' + 2y = e^{-2x} + x^2.$

20. $y'' - 3y' + 2y = xe^{-x}.$

21. $y'' + y' - 6y = x + e^{2x}.$

22. $y'' + y = \sin x + e^{-x}.$

23. $y''' - 3y'' + 3y' - y = e^x.$

24. $y'' + y = \sin^2 x.$ *Hint.* $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x.$

25. $y''' - y' = e^{2x}\sin^2 x.$

26. $y^{(5)} + 2y''' + y' = 2x + \sin x + \cos x.$ *Hint.* Solve $y^{(5)} + 2y''' + y' = 2x + e^{ix}$; see Example 21.51.

27. $y'' + y = \sin 2x \sin x.$ *Hint.* $\sin 2x\sin x = \frac{1}{2}\cos x - \frac{1}{2}\cos 3x.$

For each of the following equations, find a particular solution which satisfies the given initial conditions.

28. $y'' - 5y' - 6y = e^{3x}, \quad y(0) = 2,\ y'(0) = 1.$

29. $y'' - y' - 2y = 5\sin x, \quad y(0) = 1,\ y'(0) = -1.$

30. $y''' - 2y'' + y' = 2e^x + 2x, \quad y(0) = 0,\ y'(0) = 0,\ y''(0) = 0.$

31. $y'' + 9y = 8\cos x, \quad y(\pi/2) = -1,\ y'(\pi/2) = 1.$

32. $y'' - 5y' + 6y = e^x(2x - 3), \quad y(0) = 1,\ y'(0) = 3.$

33. $y'' - 3y' + 2y = e^{-x}, \quad y(0) = 1,\ y'(0) = -1.$

3. $y = c_1e^{-2x} + c_2e^{-x} + 2.$
4. $y = c_1e^{-2x} + c_2e^{-x} + 2e^x.$
5. $y = c_1e^{-2x} + c_2e^{-x} + \frac{1}{10}(e^{ix} - 3ie^{ix}).$
6. $y = c_1e^{-2x} + c_2e^{-x} + \frac{1}{10}(\sin x - 3\cos x).$
7. $y = c_1e^{-2x} + c_2e^{-x} + \frac{1}{10}(3\sin x + \cos x).$
8. $y = c_1e^{-2x} + c_2e^{-x} + 4 + e^x + \frac{1}{5}(\sin x - 3\cos x).$
9. $y = e^{-x/2}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right) + x^2 - 2x.$
10. $y = c_1e^{4x} + c_2e^{-2x} - xe^x - 2e^{-x}.$
11. $y = c_1 + c_2e^{3x} - \frac{e^{2x}}{5}(3\sin x + \cos x).$
12. $y = (c_1 + c_2x)e^x + (c_3 + c_4x)e^{-x} + x - \frac{\sin x}{4}.$
13. $y = c_1 + c_2e^{-x} + \frac{x^3}{3}.$
14. $y = c_1 + c_2e^{-x} + \frac{x^2}{2} - x - \frac{1}{10}(2\sin 2x + \cos 2x).$
15. $y = c_1\cos x + c_2\sin x - x(x\cos x - \sin x).$
16. $y = c_1\cos 2x + c_2\sin 2x - \frac{x}{16}(2x\cos 2x - \sin 2x).$
17. $y = c_1e^{-x} + c_2xe^{-x} + \frac{x^4e^{-x}}{12}.$
18. $y = c_1e^{-x} + c_2xe^{-x} + c_3x^2e^{-x} + \frac{x^3e^{-x}}{60}(20 - x^2).$
19. $y = c_1e^{-2x} + c_2e^{-x} + \frac{x^2}{2} - \frac{3x}{2} + \frac{7}{4} - xe^{-2x}.$
20. $y = c_1e^{2x} + c_2e^x + \frac{1}{36}(6xe^{-x} + 5e^{-x}).$
21. $y = c_1e^{-3x} + c_2e^{2x} - \frac{x}{6} - \frac{1}{36} + \frac{xe^{2x}}{5}.$
22. $y = c_1\cos x + c_2\sin x - \frac{x}{2}\cos x + \frac{1}{2}e^{-x}.$
23. $y = \left(c_1 + c_2x + c_3x^2 + \frac{x^3}{6}\right)e^x$
24. $y = c_1\cos x + c_2\sin x + \frac{1}{2} + \frac{\cos 2x}{6}.$
25. $y = c_1 + c_2e^x + c_3e^{-x} + \left(\frac{1}{12} + \frac{9\cos 2x - 7\sin 2x}{520}\right)e^{2x}.$

26. $y = c_1 + c_2\sin x + c_3\cos x + c_4x\sin x + c_5x\cos x + x^2$
$+ \frac{x^2}{8}(\cos x - \sin x).$
27. $y = c_1\cos x + c_2\sin x + \frac{x}{4}\sin x + \frac{\cos 3x}{16}.$
28. $y = \frac{10}{21}e^{6x} + \frac{45}{28}e^{-x} - \frac{1}{12}e^{3x}.$
29. $y = \frac{1}{3}e^{2x} + \frac{1}{6}e^{-x} - \frac{3}{2}\sin x + \frac{1}{2}\cos x.$
30. $y = x^2 + 4x + 4 + (x^2 - 4)e^x.$
31. $y = \cos x + \frac{2}{3}\cos 3x + \sin 3x.$
32. $y = e^{2x} + xe^x.$
33. $6y = -10e^{2x} + 15e^x + e^{-x}.$